Alternative Optimal Solutions

Consider the linear program

$$\begin{array}{llcl} \max & 6x_1 + 4x_2 & & \\ \text{s.t.} & x_1 + 4x_2 & \le & 40 \\ & 3x_1 + 2x_2 & \le & 30 \\ & 3x_1 + x_2 & \le & 24 \\ & x_1, \ x_2 & \ge & 0 \end{array}$$

or

$$\begin{array}{llcl} \min & -6x_1 - 4x_2 + 0x_3 + 0x_4 + 0x_5 & & \\ \text{s.t.} & x_1 + 4x_2 + x_3 & = & 40 \\ & 3x_1 + 2x_2 + x_4 & = & 30 \\ & 3x_1 + x_2 + x_5 & = & 24 \\ & x_1, \ x_2, \ x_3, \ x_4, \ x_5 & \ge & 0 \end{array}$$

An optimal basis for this problem consists of $\{a_1, a_2, a_5\}$ with $x^* = [4, 9, 0, 0, 3]^T$ and objective value $z^* = -60$. We compute reduced costs for the nonbasic variables using $w = [0, -2, 0]$:

$$rc_3 = 0 - [0, -2, 0]\, a_3 = 0$$
$$rc_4 = 0 - [0, -2, 0]\, a_4 = 2$$

However there is a zero reduced cost for x_3, which signals the possibility of alternative optimal solutions. Recall the expressions for z and x_B in terms of the nonbasic variables:

$$\begin{aligned} z &= -60 + 0x_3 + 2x_4 \\ x_1 &= 4 + .2x_3 - .4x_4 \\ x_2 &= 9 - .3x_3 + .1x_4 \\ x_5 &= 3 - .3x_3 + 1.1x_4 \end{aligned}$$

These relationships hold for all feasible values of x; we have simply rearranged $Ax = b$. Therefore any alternative optimal solution satisfies $z = -60 + 2x_4 = -60 \Rightarrow x_4 = 0$.

Substituting this into the above, and parameterizing with $x_3 = t$, yields

$$x_1 = 4 + .2x_3 = 4 + .2\,t$$
$$x_2 = 9 - .3x_3 = 9 - .3\,t$$
$$x_5 = 3 - .3x_3 = 3 - .3\,t$$

Of course we need all basic variables to be nonnegative so that produces the range $0 \le t \le 10$.

The set of all optimal solutions consists of the points

$$x = [4 + .2\,t\,,\, 9 - .3\,t\,,\, t\,,\, 0\,,\, 3 - .3\,t],\ \ 0 \le t \le 10$$

In particular, when $t = 0$ this gives the optimal basic feasible point

$$x = [4, 9, 0, 0, 3]$$

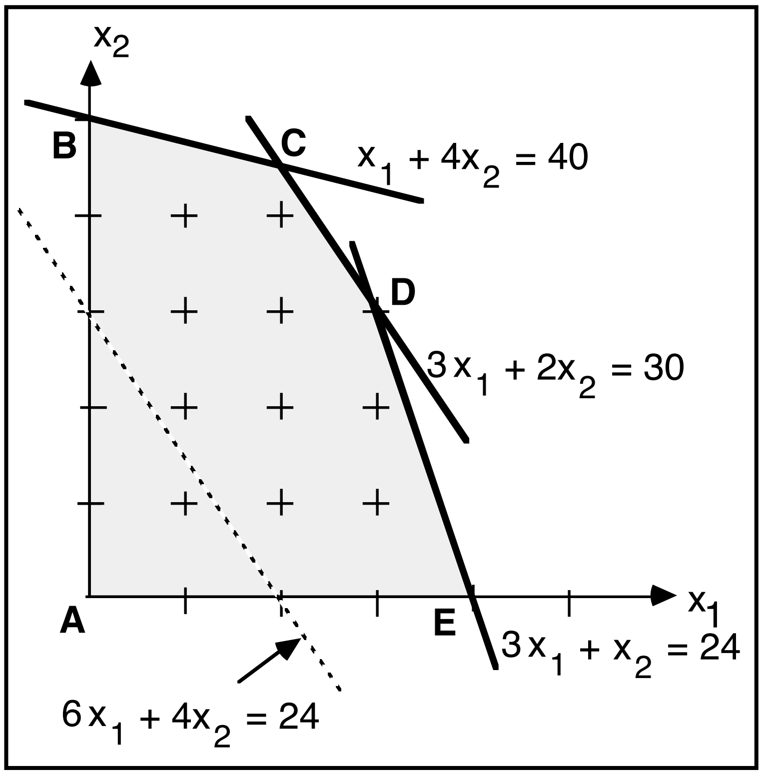
When $t = 10$ this gives the optimal basic feasible point

$$x = [6, 6, 10, 0, 0]$$

Other optimal, but not basic solutions, are obtained by using $0 < t < 10$.

Geometric Interpretation

We illustrate the feasible region of this linear program in the (x_1, x_2) plane and the set of all alternative optimal solutions.



In higher dimensions the intersection of the objective function hyperplane can be a point (unique) or an edge, a face, …

The optimal solution set is a smaller-dimensional set within the original polyhedron.